Memo December 15, 2007, from KS

This is continuation of Memos November 29 and December 5. In the notation of [S06], we have determined the limits of nested classes of the ranges of the iterations of Ψ_α in the cases $0<\alpha<1$ and $1<\alpha<2$. This time we study the case $\alpha=1$. But this case is difficult to handle, and our results are partial.

Theorem B$''$. *Let* $p(u)=u^{-2}e^{-u}$, *and* $g(t)=\int_t^\infty p(u)du$ *for* $0<t\leqslant\infty$. *Let* $t=f(s)$, $0\leqslant s<\infty$, *be defined by* $s=g(t)$, $0<t\leqslant\infty$. *Define*

$$\Phi_f(\mu)=\mathcal{L}\left(\int_0^{\infty-} f(s)dX_s^{(\mu)}\right). \tag{23}$$

Then $\mathfrak{D}(\Phi_f)$, *the domain of* Φ_f, *and* $\mathfrak{D}^0(\Phi_f)$, *the domain of absolute definability of* Φ_f, *are as follows:*

$$\begin{aligned}
\mathfrak{D}(\Phi_f)&=\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1}|x|\nu(dx)<\infty,\ \gamma=-\int_{\mathbb{R}^d}\frac{x|x|^2\nu(dx)}{1+|x|^2},\\
&\qquad \textit{and } \int_1^t s^{-1}ds\int_{|x|>s}x\nu(dx) \textit{ is convergent in } \mathbb{R}^d \textit{ as } t\to\infty\}\\
&=\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{\mathbb{R}^d}|x|\mu(dx)<\infty,\ \int_{\mathbb{R}^d}x\mu(dx)=0,\\
&\qquad \textit{and } \int_1^t s^{-1}ds\int_{|x|>s}x\nu(dx) \textit{ is convergent in } \mathbb{R}^d \textit{ as } t\to\infty\},\\
\mathfrak{D}^0(\Phi_f)&\stackrel{\text{def}}{=}\{\mu\in I(\mathbb{R}^d)\colon \int_0^\infty |C_\mu(f(s)z)|ds<\infty\}\\
&=\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1}|x|\nu(dx)<\infty,\ \gamma=-\int_{\mathbb{R}^d}\frac{x|x|^2\nu(dx)}{1+|x|^2},\\
&\qquad \textit{and } \int_1^\infty s^{-1}ds\left|\int_{|x|>s}x\nu(dx)\right|<\infty\}\\
&=\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{\mathbb{R}^d}|x|\mu(dx)<\infty,\ \int_{\mathbb{R}^d}x\mu(dx)=0,\\
&\qquad \textit{and } \int_1^\infty s^{-1}ds\left|\int_{|x|>s}x\nu(dx)\right|<\infty\}.
\end{aligned}$$

We have

$$\begin{aligned}\mathfrak{D}(\Phi_f) &\supsetneqq \mathfrak{D}^0(\Phi_f)\\ &\supsetneqq \{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1}|x|\nu(dx)<\infty,\ \gamma=-\int_{\mathbb{R}^d}\frac{x|x|^2\nu(dx)}{1+|x|^2},\\ &\qquad \textit{and } \int_1^\infty s^{-1}ds\int_{|x|>s}|x|\nu(dx)<\infty\}.\end{aligned}$$

This result is in Theorem 2.8 of [S06].

We write the triplet of $\mu\in I(\mathbb{R}^d)$ as $(A^\mu,\nu^\mu,\gamma^\mu)$ and the decomposition of the Lévy measure of $\mu\in L_\infty(\mathbb{R}^d)$ as $\Gamma^\mu(d\beta)$ and $\lambda^\mu_\beta(d\xi)$.

Theorem C$''$. *Let $f(s)$ and Φ_f be as in Theorem B$''$. Let*

$$\mathfrak{R}^m_f=\mathfrak{R}^m_f(\mathbb{R}^d)=\Phi^m_f(\mathfrak{D}(\Phi^m_f)),\qquad m=1,2,\ldots.$$

Then

$$I(\mathbb{R}^d)\supset\mathfrak{R}^1_f\supset\mathfrak{R}^2_f\supset\cdots. \tag{24}$$

Further,

(i) *If $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}^m_f$, then $\mu\in L_\infty(\mathbb{R}^d)$ and $\Gamma^\mu((0,1])=0$.*

(ii) *If $\mu\in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1])=0$, $\int_{(1,2)}(\beta-1)^{-1}\Gamma^\mu(d\beta)<\infty$, and $\int_{\mathbb{R}^d}x\mu(dx)=0$, then $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}^m_f$.*

(iii) *If $\mu\in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1])=0$, $\int_S\xi\lambda^\mu_\beta(d\xi)=0$ for Γ^μ-a.e. β, and $\gamma^\mu=0$, then $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}^m_f$.*

Proof. Assertion (i) can be proved in the same way as in the cases $0<\alpha<1$ and $1<\alpha<2$; see the proofs of Theorems C and C$'$.

Let us prove assertion (ii). Let $\mu\in L_\infty(\mathbb{R}^d)$ be such that $\Gamma^\mu((0,1])=0$, $\int_{(1,2)}(\beta-1)^{-1}\Gamma^\mu(d\beta)<\infty$ and $\int_{\mathbb{R}^d}x\mu(dx)=0$ (that is, $\gamma^\mu=-\int_{\mathbb{R}^d}x|x|^2(1+|x|^2)^{-1}\nu^\mu(dx)$). Recall that, by Lemma in Memo December 5, $\int_{\mathbb{R}^d}|x|\mu(dx)<\infty$. Define $\mu_0\in L_\infty(\mathbb{R}^d)$ in such a way that

$$\begin{aligned}A^{\mu_0}&=(\Gamma(1))^{-1}A^\mu=A^\mu,\\ \Gamma^{\mu_0}(d\beta)&=(\Gamma(\beta-1))^{-1}\Gamma^\mu(d\beta),\\ \lambda^{\mu_0}_\beta(d\xi)&=\lambda^\mu_\beta(d\xi),\\ \nu^{\mu_0}(B)&=\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\lambda^{\mu_0}_\beta(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr\qquad B\in\mathcal{B}(\mathbb{R}^d),\end{aligned}$$

$$\gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu^{\mu_0}(dx).$$

Note that, since

$$\int_{(1,2)} (\beta-1)^{-1}\Gamma^{\mu_0}(d\beta) = \int_{(1,2)} (\beta-1)^{-1}(\Gamma(\beta-1))^{-1}\Gamma^{\mu}(d\beta)$$
$$\leqslant \text{const} \int_{(1,2)} \Gamma^{\mu}(d\beta) < \infty,$$

we have $\int_{|x|>1} |x|\nu^{\mu_0}(dx) < \infty$ by Lemma in December 5. It follows from the condition on γ^{μ_0} that $\int_{\mathbb{R}^d} x\mu_0(dx) = 0$. Further,

$$\int_1^\infty s^{-1}ds \int_{|x|>s} |x|\nu^{\mu_0}(dx) = \int_1^\infty s^{-1}ds \int_{(1,2)} \Gamma^{\mu_0}(d\beta) \int_S \lambda_\beta^{\mu_0}(d\xi) \int_s^\infty r^{-\beta}dr$$
$$= \int_1^\infty s^{-1}ds \int_{(1,2)} (\beta-1)^{-1}s^{1-\beta}\Gamma^{\mu_0}(d\beta) = \int_{(1,2)} (\beta-1)^{-1}\Gamma^{\mu_0}(d\beta) \int_1^\infty s^{-\beta}ds$$
$$= \int_{(1,2)} (\beta-1)^{-2}\Gamma^{\mu_0}(d\beta) = \int_{(1,2)} (\beta-1)^{-2}(\Gamma(\beta-1))^{-1}\Gamma^{\mu}(d\beta)$$
$$\leqslant \text{const} \int_{(1,2)} (\beta-1)^{-1}\Gamma^{\mu}(d\beta) < \infty.$$

Now it follows from Theorem B$''$ that $\mu_0 \in \mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f)$. In addition, $\Phi_f(\mu_0) = \mu$, since the Lévy measure of $\Phi_f(\mu_0)$ equals ν^μ as in (20) in Memo December 5 and since the location parameter of $\Phi_f(\mu_0)$ equals γ^μ as in Step 9 in Memo December 5. Thus we see that $\mu \in \Phi_f(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f))$. Since $\int_{(1,2)}(\beta-1)^{-1}\Gamma^{\mu_0}(d\beta) < \infty$, we can similarly define $\mu_{00} \in L_\infty(\mathbb{R}^d)$ and prove that $\mu_0 = \Phi_f(\mu_{00})$. That is, $\mu_{00} \in L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f^2)$ and $\Phi_f^2(\mu_{00}) = \mu$. Repeating this procedure, we see that, for any positive integer m, there is $\mu_{(m)} \in L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f^m)$ such that $\Phi_f^m(\mu_{(m)}) = \mu$. This proves assertion (ii).

Next let us prove assertion (iii). Let $\mu \in L_\infty(\mathbb{R}^d)$ be such that $\Gamma^\mu((0,1]) = 0$, $\int_S \xi\lambda_\beta^\mu(d\xi) = 0$ for Γ^μ-a.e. β, and $\gamma^\mu = 0$. Define $\mu_0 \in L_\infty(\mathbb{R}^d)$ in the following way:

$$A^{\mu_0} = A^\mu,$$
$$\Gamma^{\mu_0}(d\beta) = (\Gamma(\beta-1))^{-1}\Gamma^\mu(d\beta),$$
$$\lambda_\beta^{\mu_0}(d\xi) = \lambda_\beta^\mu(d\xi),$$
$$\nu^{\mu_0}(B) = \int_{(1,2)} \Gamma^{\mu_0}(d\beta) \int_S \lambda_\beta^{\mu_0}(d\xi) \int_0^\infty 1_B(r\xi)r^{-\beta-1}dr \qquad B \in \mathcal{B}(\mathbb{R}^d),$$
$$\gamma^{\mu_0} = 0.$$

Then $\int_{\mathbb{R}^d}|x|\mu_0(dx)<\infty$ by Lemma in December 5 as before. We have

$$\int_{\mathbb{R}^d}\frac{x|x|^2\nu^{\mu_0}(dx)}{1+|x|^2}=\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\lambda_\beta^{\mu_0}(d\xi)\int_0^\infty\frac{r\xi r^2}{1+r^2}r^{-\beta-1}dr$$
$$=\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\xi\lambda_\beta^{\mu}(d\xi)\int_0^\infty\frac{r^{-\beta+2}}{1+r^2}dr=0.$$

Hence $\gamma^{\mu_0}=-\int_{\mathbb{R}^d}x|x|^2(1+|x|^2)^{-1}\nu^{\mu_0}(dx)$, that is, $\int_{\mathbb{R}^d}x\mu_0(dx)=0$. Further,

$$\int_{|x|>s}x\nu^{\mu_0}(dx)=\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\lambda_\beta^{\mu_0}(d\xi)\int_s^\infty r\xi r^{-\beta-1}dr$$
$$=\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\xi\lambda_\beta^{\mu}(d\xi)\int_s^\infty r^{-\beta}dr=0.$$

These together combined with Theorem B″ imply that $\mu_0\in\mathfrak{D}(\Phi_f)$. Let (A',ν',γ') denote the triplet of $\Phi_f(\mu_0)$. Then

$$A'=A^{\mu_0}=A^{\mu},$$
$$\nu'(B)=\int_{(1,2)}\Gamma(\beta-1)\Gamma^{\mu_0}(d\beta)\int_S\lambda_\beta^{\mu_0}(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr=\nu^{\mu}(B),$$
$$\gamma'=\lim_{t\to\infty}\int_0^t f(s)ds\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu^{\mu_0}(dx)$$
$$=\lim_{t\to\infty}\int_0^t f(s)ds\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\lambda_\beta^{\mu_0}(d\xi)\int_0^\infty r\xi\left(\frac{1}{1+f(s)^2r^2}-\frac{1}{1+r^2}\right)r^{-\beta-1}dr$$
$$=\lim_{t\to\infty}\int_0^t f(s)ds\int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S\xi\lambda_\beta^{\mu}(d\xi)\int_0^\infty\left(\frac{1}{1+f(s)^2r^2}-\frac{1}{1+r^2}\right)r^{-\beta}dr$$
$$=0,$$

where the integrability is checked as in page 6 of Memo November 29. Thus we have $\Phi_f(\mu_0)=\mu$. Repeating this procedure, we can find, for any m, $\mu_{(m)}\in L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f^m)$ such that $\Phi_f^m(\mu_{(m)})=\mu$. The proof of (iii) is complete.

Remarks. Let $f(s)$ and Φ_f be as in Theorems B″ and C″.

1. If μ_1 and μ_2 are in $\mathfrak{D}(\Phi_f)$, then $\mu_1*\mu_2\in\mathfrak{D}(\Phi_f)$ and $\Phi_f(\mu_1*\mu_2)=\Phi_f(\mu_1)*\Phi_f(\mu_2)$. Hence, for each positive integer m, $\mathfrak{R}_f^m$ is closed under convolution. Hence the class $\bigcap_{m=1}^\infty\mathfrak{R}_f^m$ is closed under convolution.

2. The preceding remark combined with (ii) and (iii) of Theorem C″ gives other examples of μ in $\bigcap_{m=1}^\infty\mathfrak{R}_f^m$. For instance, if, for some $\varepsilon\in(0,1]$, $\mu\in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1])=0$, $\int_S\xi\lambda_\beta^\mu(d\xi)=0$ for Γ^μ-a.e. $\beta\in(1,1+\varepsilon)$, and $\gamma^\mu=-\lim_{a\to\infty}\int_{|x|\leqslant a}\dfrac{x|x|^2\nu^\mu(dx)}{1+|x|^2}$, then $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}_f^m$.