REMARKS ON PÓLYA'S THEOREM ON CHARACTERISTIC FUNCTIONS

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1. Introduction

In 1949 G. Pólya [9] found the following fact.

Theorem 1.1. *Let $\varphi(z)$ be a real-valued continuous function on $\mathbb{R}$ satisfying $\varphi(0) = 1$, $\varphi(-z) = \varphi(z)$, and $\lim_{z\to\infty} \varphi(z) = 0$, and being convex for $z > 0$.*

(i) *Then $\varphi(z)$ is the characteristic function of a probability distribution μ on $\mathbb{R}$.*

(ii) *Furthermore one can define*

$$f(x) = \lim_{c\to\infty} \frac{1}{\pi} \int_0^c \varphi(z) \cos xz \, dz = \lim_{c\to\infty} \frac{-1}{\pi x} \int_0^c \varphi'(z) \sin xz \, dz \tag{1.1}$$

for any $x \neq 0$. This $f(x)$ is nonnegative and continuous and satisfies $f(-x) = f(x)$ on $\mathbb{R} \setminus \{0\}$. The distribution μ is absolutely continuous on $\mathbb{R}$ having this $f(x)$ as a density function.

We will give some remarks concerning proofs of Theorem 1.1 and show the following.

Theorem 1.2. *Let $\varphi(z)$ and $f(x)$ be as in Theorem 1.1. Then $\lim_{x\to\infty} f(x) = 0$. If $\varphi(z)$ is integrable, then $f(x)$ can be defined by* (1.1) *also for $x = 0$ and we have $\lim_{x\to 0} f(x) = f(0) < \infty$. If $\varphi(z)$ is not integrable, then $\lim_{x\to 0} f(x) = \infty$.*

We will make some related comments and give an application of Theorem 1.1 to a proof of a two-dimensional result of Kojo [6].

2. On proofs of Theorem 1.1

The name "Pólya's Theorem" is usually used for the assertion (i) of Theorem 1.1. There are two methods to prove (i). One is the method of polygonal lines. First, one notices that, for any $a > 0$, the tent function $\varphi(z) = (1 - a^{-1}|z|) \vee 0$ is the characteristic function of a distribution. Then one shows that, in the case

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where $\varphi(z)$ has a compact support and its graph is a polygonal line, $\varphi(z)$ is a convex combination of tent functions. Then the limiting procedure from such functions handles the general case. This method was given by Dugué and Girault [3] in 1955 and used in many books such as Feller [4] pp. 505, 509, Berg and Forst [1] pp. 28–29, and Billingsley [2] p. 363. This method proves the assertion (i) nicely but does not prove (ii). However, as Billingsley [2] mentions, we can give an example of an absolutely continuous distribution with non-integrable characteristic function by this method (in p. 363 he says "a continuous density", but the density of such a distribution diverges at the origin; see Theorem 1.2).

The second method is Fourier-analytic and proves (i) and (ii) simultaneously. It is adopted by Pólya himself. First, one proves that a function $f(x)$ can be defined by (1.1) and is nonnegative. Let us check it in detail. It follows from the assumptions on $\varphi(z)$ that it is nonnegative, decreasing[1], absolutely continuous, and expressed as

$$\varphi(z) = 1 - \int_0^z \psi(u)du, \qquad z > 0,$$

where $\psi(z)$ is a nonnegative decreasing function on $(0, \infty)$. In fact, $\varphi(z)$ is differentiable almost everywhere on $(0, \infty)$ and $\psi(z) = -\varphi'(z)$. Fix $x > 0$. Then

$$\begin{aligned}
\frac{1}{\pi}\int_0^c \varphi(z)\cos xz\, dz &= \frac{1}{\pi}\int_0^c \cos xz\, dz - \frac{1}{\pi}\int_0^c \cos xz\, dz \int_0^z \psi(u)du \\
&= \frac{\sin cx}{\pi x} - \frac{1}{\pi}\int_0^c \psi(u)du \int_u^c \cos xz\, dz \\
&= \frac{\sin cx}{\pi x} - \frac{1}{\pi}\int_0^c \psi(u)\left(\frac{\sin cx}{x} - \frac{\sin xu}{x}\right) du \\
&= \varphi(c)\frac{\sin cx}{\pi x} + \frac{1}{\pi x}\int_0^c \psi(z)\sin xz\, dz,
\end{aligned} \tag{2.1}$$

where the first term tends to 0 as $c \to \infty$. Let

$$a_j(x) = \frac{1}{\pi x}\int_0^{\pi/x} \psi\left(\frac{j\pi}{x} + z\right)\sin xz\, dz,$$

$$b(c, x) = \frac{1}{\pi x}\int_{n\pi/x}^c \psi(z)\sin xz\, dz,$$

where $n = n(c, x)$ satisfying $n\pi/x < c \leqslant (n+1)\pi/x$. Then

$$\frac{1}{\pi x}\int_0^c \psi(z)\sin xz\, dz = \sum_{j=0}^{n-1} \frac{1}{\pi x}\int_{j\pi/x}^{(j+1)\pi/x} \psi(z)\sin xz\, dz + b(c, x)$$

[1] We use *increase* and *decrease* in the wide sense allowing flatness.

$$= \sum_{j=0}^{n-1} (-1)^j a_j(x) + b(c,x).$$

Since $\psi(z)$ is nonnegative and decreasing, we have $a_j(x) \geqslant 0$ and $a_j(x) \geqslant a_{j+1}(x)$. As $c \to \infty$, n tends to ∞ and $\sum_{j=0}^{n-1}(-1)^j a_j(x)$ is convergent to a nonnegative limit function. Since $\psi(z) \to 0$ as $z \to \infty$,

$$|b(c,x)| \leqslant \frac{1}{\pi x}\int_{n\pi/x}^{c} \psi(z)dz \leqslant \frac{1}{\pi x}\int_{n\pi/x}^{(n+1)\pi/x} \psi(z)dz \to 0, \quad c \to \infty.$$

Hence we can define $f(x)$ by (1.1) for $x \neq 0$ and $f(x)$ is nonnegative. Obviously $f(-x) = f(x)$. For any choice of $0 < a < b < \infty$, the convergence in (1.1) is uniform with respect to x in $[a,b]$. Indeed, by the discussion above,

$$f(x) - \frac{1}{\pi}\int_0^c \varphi(z)\cos xz\, dz = \sum_{j=n}^{\infty} (-1)^j a_j(x) + o(1),$$

where $o(1)$ is uniform in $[a,b]$ as $c \to \infty$, and hence

$$\left| f(x) - \frac{1}{\pi}\int_0^c \varphi(z)\cos xz\, dz \right| \leqslant a_n(x) + o(1).$$

This gives the uniformity in $[a,b]$ of the convergence in (1.1), since

$$a_n(x) \leqslant \frac{1}{\pi x}\int_0^{\pi/x} \psi\left(\frac{n\pi}{x} + z\right) dz \leqslant \frac{1}{x^2}\psi\left(\frac{n\pi}{x}\right) \leqslant \frac{1}{x^2}\psi\left(c - \frac{\pi}{x}\right) \leqslant \frac{1}{a^2}\psi\left(c - \frac{\pi}{a}\right) \to 0$$

as $c \to \infty$. It follows that $f(x)$ is continuous on $(0,\infty)$. Next, if we prove

$$\int_{-\infty}^{\infty} f(x)dx = 1, \tag{2.2}$$

$$\int_{-\infty}^{\infty} e^{izx} f(x)dx = \varphi(z), \quad z \in \mathbb{R}, \tag{2.3}$$

then (i) and (ii) are true and $\mu(dx) = f(x)dx$. Concerning the proof of (2.2) and (2.3), Pólya [9] only suggests the use of Fourier theory. This is not simple, as $\varphi(z)$ is not always integrable. The proof of (i) and (ii) in Lukacs's book [8] pp. 83–84 says that here one can use the result in Titchmarsh [14] p. 16 that, if $g(z)$, $z > 0$, is decreasing to 0 as $z \to \infty$ and integrable on any finite interval, then, for any $u > 0$,

$$\frac{1}{2}(g(u+0) + g(u-0)) = \frac{2}{\pi} \lim_{\varepsilon\downarrow 0,\, R\to\infty} \int_{\varepsilon}^{R} \cos ux \left(\lim_{M\to\infty} \int_0^M g(z)\cos xz\, dz \right) dx. \tag{2.4}$$

Lukacs says that this implies (2.2) by letting $g = \varphi$ and $u = 0$, but to let $u = 0$ is erroneous. He also cites Riesz and Livingston [10] for a short proof of (2.4). If we look into [10], we can find the following better result: if $g(z)$ is a complex-valued function of bounded variation on the line $\mathbb{R}$ and $g(z) \to 0$ as $|z| \to \infty$, then

$$\begin{aligned}(2.5)\quad & (g(u+0)+g(u-0))/2 \\ & = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0,\, R\to\infty} \left(\int_{-\varepsilon}^{-R} + \int_{\varepsilon}^{R} \right) e^{-iux} \left(\lim_{M,N\to\infty} \int_{-M}^{N} g(z) e^{ixz} dz \right) dx. \quad u \in \mathbb{R},\end{aligned}$$

where the inner limit exists except possibly at $x = 0$. This result gives (2.2) and (2.3) and completes the proof of (i) and (ii) of Theorem 1. (Lukacs [8] pp. 87–88 describes also essential points of the method of polygonal lines.)

The proof of Linnik's book [7] pp. 13–14, 37–39 is also Fourier-analytic. He rigorously shows (i) and (ii) in the case where $\varphi(z)$ has a compact support, and then by approximation he gives (i) for a general $\varphi(z)$. He does not try to prove (ii) in the general case.

Let us give a simple proof of (ii), assuming the validity of (i).

Proof of (ii). We admit the existence of μ with characteristic function $\varphi(z)$. Since $\varphi(z)$ is real, μ is symmetric. By Lévy's theorem (Theorem 26.2 of [2])

$$(2.6)\qquad \mu((a,b]) = \lim_{c\to\infty} \int_a^b \frac{dx}{2\pi} \int_{-c}^{c} e^{-ixz} \varphi(z) dz = \lim_{c\to\infty} \int_a^b \frac{dx}{\pi} \int_0^c \varphi(z) \cos xz \, dz$$

for any $0 < a < b < \infty$ satisfying $\mu(\{a\}) = \mu(\{b\}) = 0$. As is explained above, we can define $f(x)$ by (1.1) for $x \neq 0$ and $f(x)$ is nonnegative and continuous. The convergence in (1.1) is uniform with respect to $x \in [a,b]$. Hence, it follows from (2.6) that

$$\mu((a,b]) = \int_a^b f(x) dx.$$

Therefore, on $\mathbb{R}\setminus\{0\}$, μ is absolutely continuous with density $f(x)$. We have $\mu(\{0\}) = 1 - \int_{-\infty}^{\infty} f(x) dx$. By the Riemann-Lebesgue theorem (Theorem 26.1 of [2]), $\varphi(z)$ tends to $\mu(\{0\})$ as $z \to \infty$. Hence $\mu(\{0\}) = 0$ by the assumption on $\varphi(z)$. This finishes the proof of (ii).

3. Proof of Theorem 1.2

Let $\varphi(z)$ and $f(x)$ be as in Theorem 1.1. Let us prove Theorem 1.2. In order to see $\lim_{x\to\infty} f(x) = 0$, recall that, for $x > 0$,

$$\left| f(x) - \frac{1}{\pi} \int_0^c \varphi(z) \cos xz \, dz \right| \leqslant a_n(x) + b(c,x) + \frac{\varphi(c)}{\pi x} \leqslant \frac{2}{x^2} \psi\left(\frac{n\pi}{x}\right) + \frac{\varphi(c)}{\pi x}$$

$$\leqslant \frac{2}{x^2}\,\psi\left(c-\frac{\pi}{x}\right)+\frac{\varphi(c)}{\pi x}.$$

Fix c and let $x \to \infty$. Then, noting that $\int_0^c \varphi(z)\cos xz\,dz \to 0$, we get $f(x) \to 0$.

If $\varphi(z)$ is integrable, then

$$f(x) = \frac{1}{\pi}\int_0^\infty \varphi(z)\cos xz\,dz$$

for $x \neq 0$ and we can define also $f(0)$ by this formula, getting $\lim_{x\to 0} f(x) = f(0) < \infty$.

Let $x > 0$. In general we have

$$\lim_{c\to\infty}\frac{1}{\pi x}\int_{2\pi/x}^{c}\psi(z)\sin xz\,dz \geqslant 0$$

by the same reason as the proof that $f(x) \geqslant 0$. Hence, it follows from (1.1) that

$$\begin{aligned} f(x) &\geqslant \frac{1}{\pi x}\int_0^{2\pi/x}\psi(z)\sin xz\,dz \\ &= \frac{1}{\pi x}\int_0^{\pi/x}\left(\psi(z)-\psi\left(\frac{\pi}{x}+z\right)\right)\sin xz\,dz. \end{aligned} \tag{3.1}$$

Choose $c > 0$ arbitrarily and let $0 < x < \pi/c$. Since $\psi(z)$ is decreasing, we have

$$f(x) \geqslant \frac{1}{\pi x}\int_0^{c}\left(\psi(z)-\psi\left(\frac{\pi}{x}+z\right)\right)\sin xz\,dz.$$

We have

$$\frac{1}{\pi x}\int_0^c \psi(z)\sin xz\,dz \to \frac{1}{\pi}\int_0^c \psi(z)z\,dz, \quad x \downarrow 0$$

since $\int_0^c \psi(z)dz = 1-\varphi(c) < \infty$, and

$$\frac{1}{\pi x}\int_0^c \psi\left(\frac{\pi}{x}+z\right)\sin xz\,dz \to 0, \quad x \downarrow 0$$

since $\psi(\infty) = 0$. Therefore,

$$\liminf_{x\downarrow 0} f(x) \geqslant \frac{1}{\pi}\int_0^c \psi(z)z\,dz = \frac{1}{\pi}\int_0^c (\varphi(z)-\varphi(c))dz.$$

Since $\varphi(c)$ decreases to 0 as $c \to \infty$,

$$\lim_{c\to\infty}\frac{1}{\pi}\int_0^c(\varphi(z)-\varphi(c))dz = \frac{1}{\pi}\int_0^\infty \varphi(z)dz$$

by the monotone convergence theorem. Now, if $\varphi(z)$ is not integrable, then the last integral is infinite and $\lim_{x\downarrow 0} f(x) = \liminf_{x\downarrow 0} f(x) = \infty$.

4. Remarks and examples

We give some simple facts related to Theorem 1.1 and some examples.

Proposition 4.1. ([1], [5]) *Let $\varphi(z)$ be a continuous function on $\mathbb{R}$ satisfying $\varphi(0) = 1$, $\varphi(-z) = \varphi(z)$, $\lim_{z\to\infty} \varphi(z) = p \in (0, 1)$, and being convex for $z > 0$. Then $\varphi(z) = \widehat{\mu}(z)$, where $\mu = p\delta_0 + (1-p)\mu_1$ with an absolutely continuous distribution μ_1.*

Proof. Let $\varphi_1(z) = (1-p)^{-1}(\varphi(z) - p)$. Then $\varphi_1(z)$ satisfies the conditions in Theorem 1.1. Hence $\varphi_1 = \widehat{\mu}_1$, where μ_1 is an absolutely continuous distribution. Let $\mu = p\delta_0 + (1-p)\mu_1$. Then $\widehat{\mu} = p + (1-p)\varphi_1 = \varphi$. $\square$

A function is said to be *log-convex* if it is positive and its logarithm is convex.

Proposition 4.2. ([5]) *Let $\varphi(z)$ be a continuous function on $\mathbb{R}$ satisfying $\varphi(0) = 1$, $\varphi(-z) = \varphi(z)$, and $0 < \varphi(z) \leqslant 1$, and being log-convex for $z > 0$. Then $\varphi(z)$ is the characteristic function of an infinitely divisible distribution on $\mathbb{R}$.*

Proof. For positive z_1 and z_2 we have

$$\log \varphi\left(\frac{z_1 + z_2}{2}\right) \leqslant \frac{\log \varphi(z_1) + \log \varphi(z_2)}{2}$$

and hence

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leqslant \sqrt{\varphi(z_1)\,\varphi(z_2)} \leqslant \frac{\varphi(z_1) + \varphi(z_2)}{2}.$$

Thus $\varphi(z)$ is convex for $z > 0$. The function $\varphi(z)$ must be decreasing for $z > 0$. Hence $\lim_{z\to\infty} \varphi(z)$ exists. Therefore, by Theorem 1.1 (i) and Proposition 4.1, $\varphi(z)$ is the characteristic function of a distribution μ. For any $t > 0$, $\varphi(z)^t$ satisfies exactly the same conditions as $\varphi(z)$. Hence $\varphi(z)^t$ is a characteristic function. It follows that μ is infinitely divisible. $\square$

We call functions $\varphi(z)$ in Theorem 1.1 and Proposition 4.1 *characteristic functions of Pólya type*, and functions $\varphi(z)$ in Proposition 4.2 *characteristic functions of log-Pólya type*. As we have seen above, log-Pólya type implies Pólya type.

It is known that there is an infinitely divisible distribution with Pólya type characteristic function which is not of log-Pólya type (Keilson and Steutel [5] p. 246).

Proposition 4.3. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with characteristic function $\widehat{\mu}(z)$ of log-Pólya type.*

(i) *Then, for any $a \in (0, \infty)$, there is an infinitely divisible distribution μ_1 with characteristic function of log-Pólya type such that $\mu_1 \neq \mu$ and $\widehat{\mu}_1(z) = \widehat{\mu}(z)$ for $z \in [-a, a]$.*

(ii) *If the given distribution* μ *is not Cauchy, then, for some* $a \in (0,\infty)$*, there is an infinitely divisible distribution* μ_2 *with characteristic function of log-Pólya type such that* $\mu_2 \neq \mu$ *and* $\widehat{\mu}_2(z) = \widehat{\mu}(z)$ *for* $z \notin [-a,a]$.

Proof. Let $\log \widehat{\mu}(z) = \psi(z)$. Given $a > 0$, we can construct a continuous function $\widetilde{\psi}(z)$ different from $\psi(z)$ such that $\widetilde{\psi}(0) = 0$, $\widetilde{\psi}(-z) = \widetilde{\psi}(z)$, $\widetilde{\psi}(z)$ is nonpositive and convex for $z > 0$, and $\widetilde{\psi}(z) = \psi(z)$ for $0 \leqslant z \leqslant a$. Indeed, if $\psi(z)$ is not linear for $a \leqslant z \leqslant b$ for some $b > a$, then let

$$\widetilde{\psi}(z) = \begin{cases} \psi(z) & \text{for } z \in [0,a] \cup [b,\infty) \\ \psi(a) + (\psi(b) - \psi(a))(z-a)/(b-a) & \text{for } z \in (a,b); \end{cases}$$

if $\psi(z) = -c(z-a) + \psi(a)$ with $c > 0$ for $z \geqslant a$, then let $0 < \alpha < 1$ and let

$$\widetilde{\psi}(z) = \begin{cases} \psi(z) & \text{for } z \in [0,a] \\ c\alpha^{-1}a(1 - a^{-\alpha}z^{\alpha}) + \psi(a) & \text{for } z \in (a,\infty). \end{cases}$$

This proves (i). In order to show (ii), choose $a > 0$ such that $\psi(z)$ is not linear for $z \in [0,a]$, and make $\widetilde{\psi}(z)$ changing the part for $z \in [0,a]$ linearly. □

Proposition 4.4. *Let* $\varphi(z)$ *be a Pólya type characteristic function. Suppose that there is* $\varepsilon > 0$ *such that* $\varphi(z)$ *is strictly convex for* $z \in (0,\varepsilon)$*. Then the density* $f(x)$ *corresponding to* $\varphi(z)$ *satisfies* $f(x) > 0$ *for all* $x \in \mathbb{R}$.

Proof. It follows from the assumption that the function $\psi(z)$ is strictly decreasing for $z \in (0,\varepsilon)$. Hence the inequality (3.1) gives the positivity of $f(x)$. □

Proposition 4.5. *Let* μ *be a distribution with Pólya type characteristic function. Then* $\int |x|\mu(dx) = \infty$*. If moreover* μ *is infinitely divisible, then its Lévy measure* ν *satisfies* $\int_{|x|>1} |x|\nu(dx) = \infty$.

Proof. If $\int |x|\mu(dx) < \infty$, then $\widehat{\mu}(z)$ is differentiable on $\mathbb{R}$. But $\widehat{\mu}(z)$ is not differentiable at $z = 0$. This proves the first assertion. The second assertion follows from the first because, for an infinitely divisible distribution μ, the properties $\int |x|\mu(dx) = \infty$ and $\int_{|x|>1} |x|\nu(dx) = \infty$ are equivalent ([11] Theorem 25.3). □

Remark 4.6. Distributions with Pólya type characteristic functions are not necessarily unimodal. Indeed, for $a > 0$, the tent function $\varphi_a(z) = (1 - a^{-1}|z|) \vee 0$ corresponds to the density $f_a(x) = (1 - \cos ax)/(\pi a x^2)$ ([2] p. 358, [4] p. 503, [8] p. 85) and this is not unimodal as it has zero points. The next simplest Pólya type characteristic function is $\varphi(z) = p\varphi_a(z) + (1-p)\varphi_b(z)$ with $0 < p < 1$ and $0 < a < b$. In this case the corresponding density function is $f(x) = pf_a(x) + (1-p)f_b(x)$. If, for example, $b = 2a$, then it is easy to see that $f(x)$ is not unimodal.

Proposition 4.7. *There is a non-unimodal infinitely divisible distribution with log-Pólya type characteristic function. Such a compound Poisson distribution exists; also such a distribution with infinite Lévy measure exists.*

Proof. Let ν be a non-unimodal distribution with Pólya type characteristic function. Let $\varphi(z) = \exp(\widehat{\nu}(z) - 1)$. Then, for any $t > 0$, $\varphi(z)^t$ is a log-Pólya type characteristic function. Let μ_t be the corresponding infinitely divisible distribution; μ_t is compound Poisson and ν is the Lévy measure of μ_1. If t is sufficiently small, then μ_t is non-unimodal. Indeed if, on the contrary, there is a sequence $t_n \downarrow 0$ such that μ_{t_n} is unimodal, then ν must be unimodal with mode 0 by virtue of Wolfe's theorem ([11] Theorem 54.1). Now choose t sufficiently small, consider $\varphi(z)^t e^{-s|z|^\alpha}$ with $0 < \alpha \leqslant 1$ and $s > 0$, and then let s be sufficiently small. This is again of log-Pólya type (see Example 4.8); the corresponding distribution is non-unimodal and its Lévy measure has infinite total mass. $\square$

Example 4.8. ([2], [4], [7], [8]) If

$$\varphi(z) = e^{-|z|^\alpha} \tag{4.1}$$

with $0 < \alpha \leqslant 1$, then φ is a characteristic function of log-Pólya type. If $1 < \alpha \leqslant 2$, then the function $\varphi(z)$ of (4.1) is a characteristic function but not of Pólya type. These are symmetric α-stable distributions.

Example 4.9. ([7]) Let

$$\varphi(z) = \frac{1}{1+|z|^\alpha} \tag{4.2}$$

with $0 < \alpha \leqslant 1$. Then $\varphi(z)$ is a characteristic function of log-Pólya type. Thus it corresponds to an infinitely divisible distribution. Indeed, for $z > 0$,

$$(\log \varphi)' = -\frac{\alpha z^{\alpha-1}}{1+z^\alpha} < 0,$$

$$(\log \varphi)'' = \frac{\alpha z^{2\alpha-2} - \alpha(\alpha-1)z^{\alpha-2}}{(1+z^\alpha)^2} > 0.$$

When $\alpha = 2$, (4.2) is the characteristic function of a symmetrized exponential distribution, which is infinitely divisible. Also when $1 < \alpha < 2$, $\varphi(z)$ of (4.2) is the characteristic function of an infinitely divisible distribution, which is proved by Linnik [7] p. 40 by complex-variable method. But the fact that (4.2) gives an infinitely divisible characteristic function for $0 < \alpha \leqslant 2$ is a consequence of the observation that it appears in subordination of the symmetric α-stable process by the Γ-subordinator (see [11] p. 203). These are called Linnik distributions or geometric stable distributions of index $0 < \alpha \leqslant 2$.

As a consequence of Theorem 1.2, the Linnik distributions of index $0 < \alpha \leqslant 1$ have densities continuous on $\mathbb{R}\setminus\{0\}$ and divergent at 0. In the case of index $1 < \alpha \leqslant 2$, they have densities bounded and continuous on $\mathbb{R}$, as their characteristic functions are integrable.

The subordination representation shows the selfdecomposability of the Linnik distributions with index $0 < \alpha \leqslant 2$ by the result of [12] p. 324. Hence they are unimodal with mode 0 by Yamazato's theorem [15].

Other interesting properties of the classes of Pólya type characteristic functions and of log-Pólya type characteristic functions are discussed in Keilson and Steutel [5] pp. 245–249 and Steutel and van Harn [13] p. 205 et seq.

We do not know whether all infinitely divisible distributions with Pólya type characteristic functions have absolutely continuous Lévy measures.

5. Kojo's result

Kojo [6] found the following interesting fact. Let $\mathbb{R}^2$ be the 2-dimensional Euclidean space, whose elements are column vectors $x = (x_1, x_2)'$, the prime denoting the transpose. Let $S = \{\xi = (\xi_1, \xi_2)' \in \mathbb{R}^2 \colon {\xi_1}^2 + {\xi_2}^2 = 1\}$, the unit circle in $\mathbb{R}^2$, and let $S_1 = \{\xi = (\xi_1, \xi_2)' \in S \colon \xi_1 > 0,\ \xi_2 > 0\}$.

Proposition 5.1. *Let $0 < \alpha \leqslant 1$. Let λ be a finite measure on S_1 with $\lambda(S_1) > 0$. Let μ be the symmetric α-stable distribution on $\mathbb{R}^2$ with characteristic function*

$$\widehat{\mu}(z_1, z_2) = \varphi(z_1, z_2) = \exp\left[-2\int_{S_1} |z_1\xi_1 + z_2\xi_2|^\alpha \lambda(d\xi)\right], \quad (z_1, z_2)' \in \mathbb{R}^2. \tag{5.1}$$

Define $\widetilde{\varphi}(z_1, z_2)$, $(z_1, z_2)' \in \mathbb{R}^2$, as follows:

$$\widetilde{\varphi}(z_1, z_2) = \begin{cases} \varphi(z_1, z_2) & \text{if } z_1 z_2 \geqslant 0 \\ \varphi(z_1, -z_2) & \text{if } z_1 z_2 < 0. \end{cases} \tag{5.2}$$

Then $\widetilde{\varphi}(z_1, z_2)$ is the characteristic function of a symmetric α-stable distribution $\widetilde{\mu}$ on $\mathbb{R}^2$ such that $\widetilde{\mu} \neq \mu$.

Remark 5.2. In the proposition above define λ^* by $\lambda^*(B) = \lambda(-B)$ for Borel sets B. Then

$$\widehat{\mu}(z_1, z_2) = \exp\left[-\int_S |z_1\xi_1 + z_2\xi_2|^\alpha (\lambda + \lambda^*)(d\xi)\right], \quad (z_1, z_2)' \in \mathbb{R}^2.$$

The measure $\lambda + \lambda^*$ is the so-called spectral measure of the symmetric α-stable distribution μ. The description of the spectral measure of $\widetilde{\mu}$ is not known.

We give a proof of Proposition 5.1, using Theorem 1.1. This is essentially the same proof as Kojo's in [6]. The idea of the proof is entirely Kojo's, although he does not use Theorem 1.1.

Proof of Proposition 5.1. If we prove that $\widetilde{\varphi}(z_1, z_2)$ is the characteristic function of some distribution $\widetilde{\mu}$ on $\mathbb{R}^2$, then $\widetilde{\mu}$ is symmetric α-stable because $\widetilde{\varphi}(-z_1, -z_2) = \widetilde{\varphi}(z_1, z_2)$ and $\widetilde{\varphi}(z_1, z_2)^t = \widetilde{\varphi}(t^{1/\alpha} z_1, t^{1/\alpha} z_2)$; moreover $\widetilde{\mu} \neq \mu$ since $\widetilde{\varphi}(z_1, z_2) \neq \varphi(z_1, z_2)$ if $z_1 z_2 < 0$. (Note that $|z_1\xi_1 - z_2\xi_2| > |z_1\xi_1 + z_2\xi_2|$ for $\xi_1 > 0$, $\xi_2 > 0$ if $z_1 z_2 < 0$.)

In order to prove that $\widetilde{\varphi}(z_1, z_2)$ is a characteristic function, we may assume that λ is concentrated at a point, that is,

$$\varphi(z_1, z_2) = \exp(-b\,|z_1\xi_1 + z_2\xi_2|^\alpha), \quad (z_1, z_2)' \in \mathbb{R}^2 \tag{5.3}$$

with $b > 0$ for some fixed $\xi_1 > 0$ and $\xi_2 > 0$. Indeed, if it is proved that $\widetilde{\varphi}(z_1, z_2)$ is a characteristic function in this case, then $\widetilde{\varphi}(z_1, z_2)$ is a characteristic function in the case where λ is concentrated to a finite number of points, and then the general case is handled by limiting procedure.

Now we assume (5.3). We have $\widetilde{\varphi}(z_1, z_2) = \widetilde{\varphi}(-z_1, z_2) = \widetilde{\varphi}(z_1, -z_2) = \widetilde{\varphi}(-z_1, -z_2)$. Let

$$\psi(z_1, z_2) = -b(z_1\xi_1 + z_2\xi_2)^\alpha \quad \text{for } z_1 > 0,\ z_2 > 0.$$

We denote the partial derivative with respect to z_j by putting the subscript j. Thus, for $z_1 > 0$ and $z_2 > 0$,

$$\psi_{j_1}(z_1, z_2) = -b\alpha(z_1\xi_1 + z_2\xi_2)^{\alpha-1}\xi_{j_1} < 0,$$
$$\psi_{j_1 j_2}(z_1, z_2) = -b\alpha(\alpha - 1)(z_1\xi_1 + z_2\xi_2)^{\alpha-2}\xi_{j_1}\xi_{j_2} \geqslant 0,$$

where $j_1, j_2 \in \{1, 2\}$. In general, for $n \geqslant 1$,

$$(-1)^n \psi_{j_1 \cdots j_n}(z_1, z_2) \geqslant 0 \quad \text{for } z_1 > 0,\ z_2 > 0,\ j_1, \cdots, j_n \in \{1, 2\}. \tag{5.4}$$

On $\{z_1 > 0,\ z_2 > 0\}$, we obtain, from $\varphi = e^{\psi}$ and from (5.4),

$$\varphi_{j_1} = e^{\psi}\psi_{j_1} < 0,$$
$$\varphi_{j_1j_2} = e^{\psi}(\psi_{j_1}\psi_{j_2} + \psi_{j_1j_2}) > 0,$$

and, in general,

$$\varphi_{j_1\cdots j_n} = e^{\psi}\sum C(k_1,\dots,k_n;n(1),\dots,n(l-1),n)\psi_{k_1\cdots k_{n(1)}}\psi_{k_{n(1)+1}\cdots k_{n(2)}}\cdots\psi_{k_{n(l-1)+1}\cdots k_n},$$

where the summation is over $1 \leqslant l \leqslant n$, $1 \leqslant n(1) < n(2) < \cdots < n(l-1) < n$, and sequences $(k_1, k_2, \dots, k_n)$ which are identical with $(j_1, \dots, j_n)$ up to the order. Here $C(k_1,\dots,k_n; n(1),\dots,n(l-1),n)$ is a nonnegative integer. If n is odd (resp. even), then each term in the summation is nonpositive (resp. nonnegative). Moreover the term $\psi_{j_1}\psi_{j_2}\cdots\psi_{j_n}$ appears with coefficient 1. Hence

$$(-1)^n\varphi_{j_1\cdots j_n}(z_1,z_2) > 0 \quad \text{for } z_1 > 0,\ z_2 > 0,\ j_1,\cdots,j_n \in \{1,2\}. \tag{5.5}$$

The function $\varphi(z_1, z_2)$ is integrable on $\{z_1 > 0,\ z_2 > 0\}$, since

$$\psi(z_1,z_2) \leqslant -2^{-1}b((z_1\xi_1)^{\alpha} + (z_2\xi_2)^{\alpha}).$$

Also $\varphi_{j_1\cdots j_n}(z_1, z_2)$ is integrable on $\{z_1 > 0\}$ for fixed $z_2 > 0$ and on $\{z_2 > 0\}$ for fixed $z_1 > 0$.

Define $f(x_1, x_2)$ for $(x_1, x_2)' \in \mathbb{R}^2$ as

$$f(x_1,x_2) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \varphi(z_1,z_2)\cos x_1z_1\cos x_2z_2\,dz_1dz_2. \tag{5.6}$$

Then $f(x_1, x_2)$ is continuous and $f(x_1, x_2) = f(-x_1, x_2) = f(x_1, -x_2) = f(-x_1, -x_2)$. If we prove

$$f(x_1,x_2) \geqslant 0, \tag{5.7}$$

$$4\int_0^\infty\int_0^\infty f(x_1,x_2)dx_1dx_2 = 1, \tag{5.8}$$

$$4\int_0^\infty\int_0^\infty f(x_1,x_2)\cos x_1z_1\cos x_2z_2\,dx_1dx_2 = \varphi(z_1,z_2) \quad \text{for } z_1 \geqslant 0,\ z_2 \geqslant 0, \tag{5.9}$$

then the proof is over, since (5.8) implies

$$\int_{-\infty}^\infty\int_{-\infty}^\infty f(x_1,x_2)dx_1dx_2 = 1 \tag{5.10}$$

and (5.9) implies

$$\begin{aligned}\int_{-\infty}^\infty\int_{-\infty}^\infty & f(x_1,x_2)e^{i(x_1z_1+x_2z_2)}dx_1dx_2 \\ &= 4\int_0^\infty\int_0^\infty f(x_1,x_2)\cos x_1z_1\cos x_2z_2\,dx_1dx_2 = \widetilde{\varphi}(z_1,z_2), \quad (z_1,z_2)' \in \mathbb{R}^2.\end{aligned} \tag{5.11}$$

So, our task is to prove (5.7)–(5.9). Let

$$f_{z_1}(x_2) = \frac{1}{\pi}\int_0^\infty \varphi(z_1,0)^{-1}\varphi(z_1,z_2)\cos x_2 z_2\, dz_2 \quad \text{for } z_1 \geqslant 0,\ x_2 \in \mathbb{R}. \tag{5.12}$$

By (5.5), $\varphi(z_1,0)^{-1}\widetilde{\varphi}(z_1,z_2)$ satisfies the conditions of Theorem 1.1 as a function of z_2. Hence, for $z_1 \geqslant 0$, $z_2 \geqslant 0$,

$$f_{z_1}(x_2) > 0, \tag{5.13}$$

$$2\int_0^\infty f_{z_1}(x_2)dx_2 = 1, \tag{5.14}$$

$$\varphi(z_1,0)^{-1}\varphi(z_1,z_2) = 2\int_0^\infty f_{z_1}(x_2)\cos x_2 z_2\, dx_2. \tag{5.15}$$

We have used Proposition 4.4 for (5.13). Let

$$g_{x_2}(z_1) = \frac{1}{\pi}\int_0^\infty \varphi(z_1,z_2)\cos x_2 z_2\, dz_2 \quad \text{for } z_1 \geqslant 0,\ x_2 \in \mathbb{R}. \tag{5.16}$$

Then $g_{x_2}(z_1) = \varphi(z_1,0)f_{z_1}(x_2) > 0$ and

$$\frac{\partial^2}{\partial z_1^2} g_{x_2}(z_1) = \frac{1}{\pi}\int_0^\infty \varphi_{11}(z_1,z_2)\cos x_2 z_2\, dz_2 \quad \text{for } z_1 > 0. \tag{5.17}$$

By virtue of (5.5), $\varphi_{11}(z_1,z_2)$ is positive and convex as a function of $z_2 > 0$. Thus, applying Theorem 1.1 to $\varphi_{11}(z_1,0)^{-1}\varphi_{11}(z_1,z_2)$, we obtain $(\partial^2/\partial z_1^2)g_{x_2}(z_1) \geqslant 0$ from (5.17). That is, $g_{x_2}(z_1)$ is convex for $z_1 > 0$. Now we can apply Theorem 1.1 to $g_{x_2}(0)^{-1}g_{x_2}(z_1)$. The property (5.7) follows from this, since

$$f(x_1,x_2) = \frac{1}{\pi}\int_0^\infty g_{x_2}(z_1)\cos x_1 z_1\, dz_1.$$

Also we have

$$2\int_0^\infty g_{x_2}(0)^{-1}f(x_1,x_2)dx_1 = 1, \tag{5.18}$$

$$g_{x_2}(0)^{-1}g_{x_2}(z_1) = 2\int_0^\infty g_{x_2}(0)^{-1}f(x_1,x_2)\cos x_1 z_1 dx_1. \tag{5.19}$$

From (5.14), (5.18), and $g_{x_2}(0) = f_0(x_2)$, we get

$$4\int_0^\infty\int_0^\infty f(x_1,x_2)dx_1dx_2 = 2\int_0^\infty f_0(x_2)dx_2 = 1,$$

that is, (5.8). From (5.15), (5.19), and $g_{x_2}(z_1) = \varphi(z_1,0)f_{z_1}(x_2)$, we get

$$4\int_0^\infty\int_0^\infty f(x_1,x_2)\cos x_1 z_1 \cos x_2 z_2\, dx_1dx_2 = 2\int_0^\infty g_{x_2}(z_1)\cos x_2 z_2\, dx_2 = \varphi(z_1,z_2),$$

that is, (5.9). This completes the proof.

References

[1] Berg, C., Forst, G. (1975) *Potential Theory on Locally Compact Abelian Groups,* Springer, New York.

[2] Billingsley, P. (1986) *Probability and Measure,* 2nd ed., Wiley, New York.

[3] Dugué, D., Girault, M. (1955) Fonctions convexes de Polya, *Publ. Inst. Statist. Univ. Paris* 4, 3–10.

[4] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications,* Vol. 2, 2nd ed., Wiley, New York.

[5] Keilson, J., Steutel, F. W. (1972) Families of infinitely divisible distributions closed under mixing and convolution, *Ann. Math. Statist.* 43, 242–250.

[6] Kojo, K. (2000) On two-dimensional symmetric stable distributions whose projections in some range of directions are identical, *Research Reports, Inst. Statist. Math.* 127, 27–31. (In Japanese)

[7] Linnik, Yu. V. (1964) *Decomposition of Probability Distributions,* Oliver and Boyd, Edinburgh.

[8] Lukacs, E. (1970) *Characteristic Functions,* 2nd ed., Griffin, London.

[9] Pólya, G. (1949) Remarks on characteristic functions, *Proc. Berkeley Symp. on Math. Statist and Probab.*, 115–123.

[10] Riesz, M., Livingston, A. E. (1955) A short proof of a classical theorem in the theory of Fourier integrals, *Amer. Math. Monthly*, 62, 434–437.

[11] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge.

[12] Sato, K. (2001) Subordination and self-decomposability, *Statist. Probab. Letters* 54, 317–324.

[13] Steutel, F.W., van Harn, K. (2004) *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, New York.

[14] Titchmarsh, E. C. (1948) *Introduction to the Theory of Fourier Integrals,* 2nd ed., Clarendon Press, Oxford.

[15] Yamazato, M. (1978) Unimodality of infinitely divisible distribution functions of class L, *Ann. Probab.* 6, 523–531.

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